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LETTER TO THE EDITOR

Finite-size scaling and conformal invariance in a self-dual quantum $Z(5)$ model

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Abstract. We consider the critical behaviour of a particular set of one-dimensional self-dual models with $Z(N)$ symmetry ($N \leq 5$). The critical indices are evaluated using standard finite-size scaling and by exploiting their relations with the mass gap amplitudes predicted by conformal invariance. Our results strongly suggest that the recently introduced $Z(N)$ quantum field theory is the underlying field theory for these statistical mechanics models.

In recent years much attention has been devoted to the study of two-dimensional lattice statistical models invariant under $Z(N)$ global transformations (Elitzur *et al* 1979, Alcaraz and Köberle 1980, 1981, Cardy 1980). These spin models are defined in terms of $Z(N)$ spin variables:

$$S(\boldsymbol{r}) = \exp(\mathrm{i}2\pi/N)n(\boldsymbol{r}) \qquad (n(\boldsymbol{r}) = 0, 1, \ldots, N-1)$$

located at the lattice sites $\boldsymbol{r}$.

Since the work of Kramers and Wannier (1941), duality has proved to be a powerful tool for examining the behaviour of systems undergoing phase transitions. The most general self-dual $Z(N)$ model with only next-nearest-neighbour interactions, on the square lattice, is defined by the Hamiltonian

$$H = \sum_{i,j} [H_1(n(i,j) - n(i+1,j)) + H_{-1}(n(i,j) - n(i,j+1))] \tag{1a}$$

where

$$H_k(n) = -\sum_{m=1}^{\bar{N}} J_{km}\left[\cos\left(\frac{2\pi}{N}mn\right) - 1\right] \qquad k = -1, 1 \tag{1b}$$

and $\bar{N}$ is the integer part of $N/2$ and J_{km}; $k = -1, 1$, $m = 1, 2, \ldots, \bar{N}$ are the coupling constants in the X and Y directions respectively. The corresponding Boltzmann weights are given by

$$X_n^{(k)} \equiv \exp(-\beta H_k(n)) \qquad k = -1, 1 \qquad n = 0, 1, \ldots, N-1 \tag{2}$$

and under the duality transformation these weights are transformed to (Alcaraz and Köberle 1980, 1981)

$$\tilde{X}_n^{(k)} = \left[\sum_{m=0}^{N-1} \exp\left(\frac{\mathrm{i}2\pi mn}{N}\right) X_m^{(-k)}\right]\left(\sum_{m=0}^{N-1} X_m^{(-k)}\right)^{-1}. \tag{3}$$

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The self-dual subspace, fixed under the duality transformation ($\tilde{X}_n^{(k)} = X_n^{(k)}$, $k = -1, 1$; $n = 0, 1, \ldots, N-1$), is a line for $N = 2, 3$, a plane for $N = 4, 5$, etc, and coincides with the critical surface in the regions of the parameter space where the transition is unique.

Fateev and Zamolodchikov (1982), by looking for possible solutions of the star-triangle relations for self-dual $Z(N)$ models, were able to find the free energy per particle for a particular family of $Z(N)$ spin models on the square lattice. Their solution corresponds to the Boltzmann weights

$$X_0^{(k)} = 1 \qquad X_n^{(1)} = f_n(\alpha) \qquad X_n^{(2)} = f_n(\pi - \alpha) \qquad k = -1, 1$$
$$n = 1, \ldots, N-1 \tag{4a}$$

where

$$f_n(\alpha) = \prod_{k=0}^{n-1} \sin\left(\frac{\pi k}{N} + \frac{\alpha}{2N}\right)\left[\sin\left(\frac{\pi(k+1)}{N} - \frac{\alpha}{2N}\right)\right]^{-1} \tag{4b}$$

and α is an arbitrary constant that fixes the anisotropy of the model.

More recently the same authors (Zamolodchikov and Fateev 1985) have constructed and made important predictions for a self-dual quantum field theory possessing $Z(N)$ invariance in $(1+1)$ dimensions. The conformal anomaly or central charge of their Virasoro algebra is given by

$$c = 2(N-1)/(N+2). \tag{5a}$$

These theories have $(N-1)$ fields (order parameters) with anomalous dimensions

$$2d_n = n(N-n)/N(N+2) \qquad n = 1, 2, \ldots, N-1 \tag{5b}$$

and $(N-1)$ dual fields (disorder parameters) with the same dimensions as in (5*b*), due to the self-dual behaviour of the theory. From (5*b*) we can see that $d_{N-n} = d_n$, $n = 1, \ldots, N-1$ so that we have only $\tilde{N}$ operators with distinct dimensions. There are also $\tilde{N}$ $Z(N)$ neutral fields with dimension

$$2D_n = 2n(n+1)/(N+2) \qquad n = 1, 2, \ldots, \tilde{N}. \tag{5c}$$

Zamolodchikov and Fateev (1985) conjectured that the statistical mechanics model with the Boltzmann weights (4) is critical and conformally invariant, the essential ingredients of its underlying field theory being given by (5*a*)-(5*c*); the $Z(N)$ charged ('magnetic') and neutral ('thermal') exponents are related to (5*b*) and (5*c*) respectively†.

The purpose of this letter is to test the above conjecture for the cases where $N \leq 5$ by using finite-size lattices. The cases $N = 6$ and 7 will be treated in a more extensive report. In order to perform a finite-size scaling (FSS) analysis of the model defined by the Hamiltonian (1) at the couplings given in (4) we need, as usual, to calculate the leading eigenvalues of the associated transfer matrix. This is generally difficult even for small lattices (size L) because the associated Hilbert space has dimension N^L and the transfer matrix is dense. However, more recently, it has been shown (Alcaraz and Lima Santos 1986) that the family of one-dimensional quantum $Z(N)$ models governed by the Hamiltonian

$$H_N = -\sum_{i=-\infty}^{\infty} \sum_{n=1}^{N-1} [S^n(i)S^{+n}(i+1) + R^n(i)]/\sin(\pi n/N) \tag{6}$$

† It is interesting to remark that the above dimensions (5*b*) and (5*c*) correspond exactly to the exponents of the antiferromagnetic critical points of the RSOS model (Andrews *et al* 1984, Huse 1984).

has an infinite number of local and non-local conservation laws. In (6), $S(i)$, $R(i)$ are quantum operators that satisfy the $Z(N)$ algebra

$$[S(i), R(j)] = [S(i), S(j)] = [R(i), R(j)] = 0 \qquad i \neq j$$

$$S(i)R(i) = \exp(\mathrm{i}2\pi/N)R(i)S(i) \qquad R^N(i) = S^N(i) = 1.$$

Moreover the generator of the infinite set of charges corresponds to the diagonal-to-diagonal transfer matrix T_D of the model (1) with the couplings (4), the first charge being the Hamiltonian (6), i.e. $[T_\mathrm{D}, H] = 0$. Consequently we can, in an equivalent way, study the Hamiltonians (6), which are sparse matrices, instead of the Euclidean models given by (1) and (4). The ground state energy per particle for the infinite system has also been evaluated (Alcaraz and Lima Santos 1986)

$$E_0 = -N \int_0^\infty \frac{\sinh(\frac{1}{2}\pi x)\sinh[\frac{1}{2}\pi x(N-1)]}{\cosh^2(\frac{1}{2}\pi x)\cosh(\frac{1}{2}\pi N x)}\,\mathrm{d}x - \sum_{n=1}^{N-1} \frac{1}{\sin(\pi n/N)}. \tag{7}$$

The conjecture for $N=2$ ($N=3$) is easily verified because the models (4) and (6) correspond to the Euclidean and Hamiltonian versions of the critical Ising (3-state Potts) model. The central charge c as well as the anomalous dimensions X_m for the order (or disorder) operator and X_ε for the energy operator are exactly given by the predicted values (5) (Belavin *et al* 1984a, b).

The $Z(4)$ case can be better analysed by replacing the operators $S(i)$, $R(i)$ in (6) in terms of two Pauli matrices $\sigma^z(i)$, $\sigma^x(i)$; $\tau^z(i)$, $\tau^x(i)$ at each lattice point

$$H_4 = -\sum_i [\sqrt{2}(\sigma^x(i) + \tau^x(i)) + \sigma^x(i)\tau^x(i) + \sqrt{2}(\sigma^z(i)\sigma^z(i+1)$$
$$+ \tau^z(i)\tau^z(i+1)) + \sigma^z(i)\sigma^z(i+1)\tau^z(i)\tau^z(i+1)].$$

This Hamiltonian is critical, being a particular point $\beta = \frac{1}{2}\sqrt{2}$ (see Kohmoto *et al* 1981, Alcaraz and Drugowich de Felício 1984) in the critical line of the quantum Ashkin–Teller model with correlation length, magnetic and electric exponents given by

$$\nu = \tfrac{3}{4} \qquad \gamma_m = \tfrac{21}{16} \qquad \gamma_p = \tfrac{5}{4}$$

respectively. These exponents give the following dimensions for the energy, magnetic and electric operators: $X_\varepsilon = d - 1/\nu = \frac{2}{3}$, $X_m = \frac{1}{2}(d - \gamma_m/\nu) = \frac{1}{8}$ and $X_p = \frac{1}{2}(d - \gamma_p/\nu) = \frac{1}{6}$, which reproduces (5*b*) and (5*c*) for $N=4$. The second neutral operator is marginal $X_{\varepsilon\varepsilon} = 2$ and probably corresponds to the marginal operator (four-spin couplings) of the eight-vertex model (Kadanoff and Wegner 1971). To complete the $Z(4)$ case we mention that the equivalence between the quantum Ashkin–Teller model and the *XXZ* model (see e.g. Kohmoto *et al* 1981) implies that this model must also have $c=1$ (Blöte *et al* 1986) in agreement with (5*a*).

For the rest of this letter we will concentrate on the $Z(5)$ case whose Hamiltonian, for a lattice of size L, is

$$H_5 = -\sum_{i=-L/2}^{L/2} \{\lambda[(R(i) + R^+(i))/\sin(\pi/5) + (R^2(i) + R^{+2}(i))/\sin(2\pi/5)]$$
$$+ (S(i)S^+(i+1) + S^+(i)S(i+1))/\sin(\pi/5)$$
$$+ (S^2(i)S^{+2}(i+1) + S^{+2}(i)S^2(i+1))/\sin(2\pi/5)\} \tag{8}$$

where the coupling λ, which plays the role of temperature, has been introduced†. The above Hamiltonian, with periodic boundary conditions, commutes with the $Z(5)$ charge operator

$$\exp(i2\pi Q/5) = \prod_{i=-L/2}^{L/2} R(i)$$

and consequently in the R basis the Hilbert space is separated into five disjoint sectors labelled by the eigenvalues of Q $(q = 0, 1, \ldots, 4)$. The ground state $E_0^{(0)}(\lambda)$ is in the $Q = 0$ sector and the sectors with $Q = q$ and $Q = (5 - q)$ are degenerate. These sectors can be further block diagonalised according to the eigenvalues of the translation operator (linear momentum operator). All the numerical calculation of eigenvalues in this letter were performed using the Lanczos method (Hamer and Barber 1981a, b, Roomany *et al* 1980) and periodic boundary conditions in (8).

We initially apply FSS theory (Barber 1983) in order to calculate the critical temperature λ_c and the thermal exponents ν and α. From the lowest eigenenergies $E_0^{(q)}(\lambda, L)$ of sectors $q = 0, 1, 2$ we define the two mass gaps

$$\Lambda_L^{(q)} \equiv E_0^{(q)}(\lambda, L) - E_0^{(0)}(\lambda, L) \qquad q = 1, 2. \tag{9}$$

According to FSS theory the critical temperature λ_c may be estimated by the limiting value $(L \to \infty)$ of the sequence $\lambda_c^{(i)}(L)$ obtained by solving

$$R_L^{(q)} = L\Lambda_L^{(q)}(\lambda_c^{(q)})/(L-1)\Lambda_{L-1}^{(q)}(\lambda_c^{(q)}) = 1 \qquad q = 1, 2.$$

In table 1 both sequences are shown for $L = 3$-8. Using VBS approximants (Vanden *et al* 1979, Hamer and Barber 1981b) the extrapolated values are $\lambda_c^{(1)}(\infty) = 1.000\,00 \pm 0.000\,02$ and $\lambda_c^{(2)}(\infty) = 1.000\,02 \pm 0.000\,02$. These values are consistent with the existence of a unique critical point at $\lambda = \lambda_c = 1$ which corroborates the conjecture that models (4) and (6) are critical. The exponents ν and α may be calculated using the Callan–Symanzik β function (Hamer *et al* 1979)

$$\beta_L^{(q)}(\lambda) = -\Lambda_L^{(q)}(\lambda)/[\Lambda_L^{(q)}(\lambda) - 2\lambda\partial\Lambda_L^{(q)}/\partial\lambda]$$

and the analogue of the specific heat per site

$$C_L(\lambda) = -(\lambda^2/L)\partial^2 E_0^{(0)}/\partial\lambda^2.$$

Table 1. Sequences of estimators for the critical temperature of the Hamiltonian (8). $\lambda_c^{(q)}(L)$ are obtained by using sectors 0 and q.

L	$\lambda_c^{(1)}(L)$	$\lambda_c^{(2)}(L)$
3	1.057 017	1.037 053
4	1.021 479	1.011 860
5	1.011 018	1.005 302
6	1.006 644	1.002 847
7	1.004 425	1.001 718
8	1.003 153	1.001 125

† The Hamiltonian (8) may also be obtained in the time-continuum limit $(\alpha \to 0)$ (Fradkin and Susskind 1978) of (1) $(N = 5)$ around the point (4).

In table 2 we show, at $\lambda = 1$, the value of these functions together with the mass gaps (9). From FSS theory we expect (Barber 1983) $\beta_L^{(q)}(\lambda_c) \sim L^{1/\nu}$, $C_L(\lambda_c) \sim L^{\alpha/\nu}$ as $L \to \infty$ from which we extract, using VBS approximants, $1/\nu = 1.415 \pm 0.005$ and $\alpha/\nu = 0.82 \pm 0.02$ which produces a value of $X_\varepsilon = 0.575 \pm 0.005$ in close agreement with the prediction (5*c*).

Statistical mechanics systems at criticality are believed to be conformally invariant (Polyakov 1970). In two dimensions this symmetry has many important implications (for a recent review see Cardy (1986b)). In particular Cardy (1984, 1986a) has derived a set of remarkable relations between the eigenspectrum of the statistical system in a strip of finite width and the anomalous dimensions of the operators describing the critical behaviour of the infinite system.

The relevant results, for our purposes, may be stated as follows. To each primary operator $\varnothing$, the anomalous dimension $X_\varnothing$ and spin $s_\varnothing$ in the operator algebra of the infinite system there exists a set of states in the quantum Hamiltonian, in a periodic chain of L sites, whose energy and momentum, at $\lambda = \lambda_c$, are given by

$$E_{n,n'} = E_0^{(0)} + \frac{2\pi}{L}\zeta(X_\varnothing + n + n') + O(L^{-1}) \qquad n, n' = 0, 1, 2, \ldots \tag{10a}$$

$$P_{n,n'} = \frac{2\pi}{L}(s_\varnothing + n - n') \qquad n, n' = 0, 1, 2, \ldots \tag{10b}$$

respectively as $L \to \infty$. The constant ζ is unity in the transfer matrix formalism, but is model dependent for the Hamiltonian system (see for example Alcaraz and Drugowich de Felício 1984, Gehlen *et al* 1986).

Before we apply the relations (10) to the Hamiltoanian (8) let us denote by $E_n^{(q)}(k)$ the energy corresponding to its nth excited state in the sector with charge $Q = q$ and momentum k. The $Z(5)$ neutral operators are related to states in the $q = 0$ sector, while the order (or disorder) operators are related to the $q \neq 0$ sectors. The energy operator is the first neutral operator and its anomalous dimension is estimated by

$$G_L^{(0)}(1) \equiv E_1^{(0)}(0) - E_0^{(0)}(0) = \frac{2\pi\zeta}{L} X_\varepsilon + O(L^{-1}) \tag{11a}$$

while the dimension $X_{\varepsilon\varepsilon}$ for the second neutral operator is estimated by

$$G_L^{(0)}(2) \equiv E_2^{(0)}(0) - E_0^{(0)}(0) = \frac{2\pi\zeta}{L} X_{\varepsilon\varepsilon} + O(L^{-1}). \tag{11b}$$

Table 2. Finite-lattice results for the $Z(5)$ model. Listed are the values at $\lambda = \lambda_c = 1$ of the mass gap $\Lambda_L^{(q)}$ $(q = 1, 2)$, the β functions $\beta^{(q)}$ $(q = 1, 2)$ and the specific heat C_L.

L	$\Lambda_L^{(1)}(1)$	$\Lambda_L^{(2)}(1)$	$\beta_L^{(1)}(1)$	$\beta_L^{(2)}(1)$	$C_L(1)$
2	2.222 415 179	2.951 764 728	0.302 927 480	0.316 275 389	1.727 5464
3	1.378 801 132	1.881 879 374	0.166 548 927	0.176 238 947	2.817 1660
4	1.001 817 610	1.388 298 182	0.110 328 601	0.117 553 553	3.766 0438
5	0.787 041 336	1.101 593 945	0.080 414 545	0.086 050 594	4.642 7419
6	0.648 098 818	0.913 634 271	0.062 166 443	0.066 724 026	5.472 4798
7	0.550 802 563	0.780 710 922	0.050 027 540	0.053 816 064	6.268 0440
8	0.478 856 835	0.681 666 394	0.041 451 272	0.044 668 541	7.037 0613

The charge q operators, with dimensions $X_m^{(q)}(q=1,,\ldots,4)$, which govern the correlations $\langle S^q(i)S^{+q}(i+n)\rangle$ are estimated by the relation

$$G_L^{(q)}(1) \equiv E_0^{(q)}(0) - E_0^{(0)}(0) = \frac{2\pi\zeta}{L} X_m^{(q)} \qquad q=1,\ldots,4. \tag{11c}$$

Due to the generacy of the sectors mentioned before, $X_m^{(q)} = X_m^{(N-q)}$ (with $q=1,\ldots,4$), in agreement with (5*b*). The constant ζ may be extracted from the difference in energy of two successive states related to the same primary operator (Gehlen *et al* 1986, Alcaraz and Barber 1987); for example we can use

$$Z_L \equiv E_0^{(1)}\left(\frac{2\pi}{L}\right) - E_0^{(1)}(0) = \frac{2\pi\zeta}{L} + \mathrm{O}(L^{-1}). \tag{11d}$$

In table 3 we give our estimators (11*a*–*c*) for lattices up to size $L=9$. The extrapolation of these sequences, using the alternate ε algorithm (Hamer and Barber 1981b) gives the values

$$X_\varepsilon = 0.572 \pm 0.002 \qquad X_{\varepsilon\varepsilon} = 1.73 \pm 0.02$$

for the neutral operators and

$$X_m^{(1)} = X_m^{(4)} = 0.1143 \pm 0.0001 \qquad X_m^{(2)} = X_m^{(3)} = 0.1712 \pm 0.0001$$

for the magnetic ones. These values are in close agreement with the predicted values given by (5*b*, *c*):

$$X_{\varepsilon\varepsilon} = \tfrac{12}{7} \approx 1.714\,32 \qquad X_m^{(1)} = \tfrac{4}{35} \approx 0.114\,29 \qquad X_m^{(2)} = \tfrac{6}{35} \approx 0.171\,43.$$

We emphasise that the agreement is reasonable even for $X_{\varepsilon\varepsilon}$, which corresponds to an irrelevant operator.

Table 3. Ratio of mass-gap amplitudes for the $Z(5)$ model; see equations (11*a*–*d*).

L	LZ_L	$G_L^{(0)}(1)/Z_L$	$G_L^{(0)}(2)/Z_L$	$G_L^{(1)}(1)/Z_L$	$G_L^{(2)}(1)/Z_L$
2	21.194 8214	1.172 6680	1.901 6161	0.209 7130	0.278 5364
3	27.063 3078	0.881 0805	1.714 6664	0.152 8418	0.208 6086
4	29.239 7287	0.787 1071	1.695 1805	0.137 0488	0.189 9194
5	30.243 1681	0.740 9454	1.699 4783	0.130 1189	0.182 1228
6	30.774 4725	0.713 3835	1.707 3789	0.126 3577	0.178 1283
7	31.083 1304	0.694 9456	1.714 8301	0.124 0421	0.175 8181
8	31.274 5378	0.681 6602	1.721 0695	0.122 4911	0.174 3697
9	31.399 0076	0.671 5755	1.726 1138	0.121 3876	0.173 4077

In additional to the predictions (11) conformal invariance also predicts (Blöte *et al* 1986, Affleck 1986) that the ground state energy at criticality should behave as

$$E_0^{(0)}(0)/L = e_0 - \tfrac{1}{6}\pi c\zeta/L^2 + \mathrm{O}(L^{-2}) \qquad L\to\infty. \tag{12}$$

Here c is the central charge of the conformal class governing the transition in the infinite system and e_0 is the ground state energy per site in the infinite lattice limit which for the Hamiltonian (8) can be calculated exactly by using (7): $e_0 = -6.431\,029\,721\,005\ldots$.

The conformal anomaly c can be extracted by extrapolating the sequence

$$c_L \equiv -12(E_0^{(0)}(0) - Le_0)/Z_L. \tag{13}$$

Table 4. Finite-size sequence for the extrapolation of the conformal anomaly c for the $Z(5)$ model.

L	$-E_0^{(0)}(0)/L$	$-12(E_0^{(0)}(0)-e_0L)/Z_L$	$-12(E_0^{(0)}(0)-e_0L)L/10\pi$
2	7.265 085	1.888 887	1.274 342
3	6.781 381	1.398 128	1.204 420
4	6.624 093	1.267 735	1.179 918
5	6.553 365	1.213 515	1.168 214
6	6.515 504	1.185 820	1.161 608
7	6.492 871	1.169 858	1.157 465
8	6.478 263	1.159 890	1.154 670
9	6.468 285	1.153 301	1.152 681

In table 4 we exhibit this sequence. The VBS approximants give an extrapolated value of $c = 1.13(5)$ which is close to the prediction ($5a$). One of the major error sources in this estimate of c concerns the evaluation of the constant ζ (Alcaraz and Barber 1986). However the value of ζ can be conjectured by returning to the general Hamiltonian (6). For $N=2$ ζ is exactly 2 while previous finite-size calculations indicate $\zeta=3$ for $N=3$ (Gehlen *et al* 1986) and $\zeta=4$ for $N=4$ (Alcaraz and Drugowich de Felício 1984). These facts and the numbers of table 3, for the case $N=5$, suggest that $\zeta=N$ for any $N\geqslant 2$. In table 4 we also show the sequence (13) with Z_L fixed to the conjectured value of $10\pi/L$ which gives the extrapolated value $c=1.142(9)$ in excellent agreement with the predicted value $c=\frac{8}{7}\approx 1.142\,85$ given by ($5a$).

In summary, by using FSS we have shown that the Hamiltonian (6) for $N\leqslant 5$ is critical. Exploring the finite-size implications of the conformal invariance of the infinite critical system our results strongly indicate that the $Z(N)$ self-dual quantum field theory introduced by Zamolodchikov and Fateev (1985) is the underlying field theory for the statistical models (4) and (6). To conclude we would like to mention we believe (Fateev and Zamolodchikov 1982) that for $N\geqslant 5$ the models (6) and (8) describe the bifurcation points in the phase diagram of the general model (1), where a soft phase originates (Alcaraz and Köberle 1980).

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